Waves and Landau Damping in Collisionless Plasma.

→ Phase space flow incompressible (Liouville Thm.)

→ Derive Vlasov Eqn. from:

- Liouville Eqn.
- $N = \sum_i \delta(\underline{x}-\underline{x}_i)\,\delta(\underline{v}-\underline{v}_i)$ — Klimontovich Eqn.

  "peas"
- hierarchy, with $f(\underline{X}_1, \underline{X}_2, t) =$

  $f(\underline{X}_1,t)\,f(\underline{X}_2,t) + g(\underline{X}_1,\underline{X}_2,t)$ ← "crushed pea soup"

  and $1/n\lambda_D^3 \ll 1 \Rightarrow g \ll f^2$ etc.

(Return in Fluctuations Discussion)

IV.) Collective Response in Collisionless Plasma

→ Waves in Vlasov Plasma (1D)

- $\omega, kv \gg \nu \Rightarrow$

$$f = \langle f \rangle + \tilde{f}$$

$$\langle f \rangle = \left(1/\sqrt{2\pi}\,v_{Th}\right) \exp\left(-v^2/2v_{Th}^2\right)$$

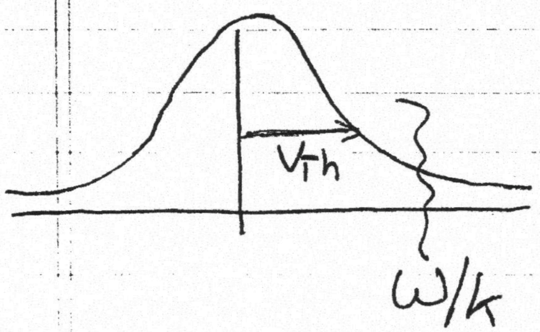
(Maxwellian)

i.e. $\langle f \rangle$ established on long-time scale

- seek contact with Langmuir Wave (ions stationary)

$\Rightarrow \omega > k v_{th}$

(Heuristic)



Then, linearize:

$$\frac{\partial \tilde{f}}{\partial t} + v \frac{\partial \tilde{f}}{\partial x} = -\frac{q}{m} \tilde{E} \frac{\partial \langle f \rangle}{\partial v}$$

$$\nabla^2 \tilde{\phi} = -4\pi n_0 q \int \tilde{f}\, dv$$

$$f = \sum_{k,\omega} f_{k,\omega}\, e^{i(kx - \omega t)}$$

$$\Rightarrow -i(\omega - kv)\tilde{f}_{k,\omega} = \frac{q}{m} ik \phi_{k,\omega}\, \partial\langle f\rangle/\partial v$$

$$+ k^2 \tilde{\phi}_{k,\omega} = 4\pi n_0 q \int \tilde{f}_{k,\omega}\, dv$$

$$\tilde{f}_{k,\omega} = -k \frac{q}{m} \frac{\phi_{k,\omega}\, \partial\langle f\rangle/\partial v}{(\omega - kv)}$$

so $$k^2 \tilde{\phi}_{k,\omega} = -\omega_p^2 k \int dv\, \frac{\partial\langle f\rangle/\partial v}{(\omega - kv)}\, \tilde{\phi}_{k,\omega}$$

Thus, $\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int dv \frac{\partial\langle f\rangle/\partial v}{(\omega - kv)}$

– dielectric function for Vlasov Plasma

$\boxed{\text{? How Handle Pole at } \omega = kv \text{ ?}}$

– Recall V. E. derived in limit $\gamma \to 0$

Concepts
– wave-particle resonance
– collisionless damping

$$1/\omega - kv = \lim_{\epsilon \to 0} 1/\omega - kv + i\epsilon$$

– Alternatively, causality requires: $\tilde{\phi} \underset{t \to -\infty}{\to} 0$

$$\phi \sim e^{-i\omega t} \Rightarrow \phi \sim e^{-i(\omega + i\epsilon)t}$$

(i.e. formally IVP)

$\therefore$

$$1/\omega - kv = \lim_{\epsilon \to 0} 1/\omega - kv + i\epsilon$$

$$= \frac{\mathcal{P}}{\omega - kv} - i\pi\delta(\omega - kv)$$

(Plemelj Formulae)

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int dv \, \frac{\partial\langle f\rangle/\partial v}{\omega - kv}$$

$$= 1 + \frac{\omega_p^2}{k}\int dv \, \frac{P}{\omega - kv}\frac{\partial\langle f\rangle}{\partial v}$$

$$- i\pi \frac{\omega_p^2}{k|k|}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k} \rightarrow \text{physical content!?}$$

i.e.

$$\delta(\omega - kv) = \frac{1}{|k|}\delta(v - \omega/k)$$

Further: $\frac{\partial\langle f\rangle}{\partial v} = -\frac{v}{v_{Th}}\langle f\rangle$

$$kv_{Th} < \omega \Rightarrow \frac{P}{\omega - kv} = \frac{1}{\omega}\left(1 + \frac{kv}{\omega} + \left(\frac{kv}{\omega}\right)^2 + \left(\frac{kv}{\omega}\right)^3 + \dots\right)$$

$$\epsilon_r(k,\omega) = 1 - \frac{\omega_p^2}{k v_{Th}^2}\int \frac{\langle f\rangle v}{\omega}\left(1 + \frac{kv}{\omega} + \left(\frac{kv}{\omega}\right)^2 + \left(\frac{kv}{\omega}\right)^3 + \dots\right)$$

$$= 1 - \frac{\omega_p^2}{\omega^2} - 3\omega_p^2\frac{v_{Th}^2 k^2}{\omega^4}$$

N.B. $$\langle x^4 \rangle = \int dx\; x^4\, e^{-x^2/2}$$

$$= 4 \frac{\partial^2}{\partial \alpha^2}\Big|_{\alpha=1} \int dx\; e^{-\alpha x^2/2}$$

$$= 4 \frac{\partial^2}{\partial \alpha^2}\Big|_{\alpha=1} \left(\text{[redacted]}\; 1/\sqrt{\alpha} \right)$$

$$= \cancel{4}\, \frac{3}{\cancel{4}} \qquad (\pi \text{ via normalization})$$

→ "3" appears from moments of Gaussian

→ Moments replace/underly equation of state.

$$\epsilon_r(k,\omega) = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2 V_{Th}^2}{\omega^2}\right)$$

so

$$\epsilon = \epsilon_R + i\,\epsilon_{IM}$$

$$\epsilon_R = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2 V_{Th}^2}{\omega^2}\right)$$

$$\epsilon_{IM} = -\pi\frac{\omega_p^2}{k|k|}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}$$

$\rightarrow \epsilon_R = 0 \Rightarrow$ collective Resonance / Wave

- as ϵ derived via $(kv/\omega) < 1$ expansion, need determine $\omega(k)$ iteratively

$$\epsilon_r = 0 = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2 V_{Th}^2}{\omega^2}\right)$$

Lowest order: $\omega^{(0)} = \omega_p$

$$\rightarrow \quad \epsilon_r = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2 V_{Th}^2}{\omega_p^2}\right)$$

$\therefore \quad \omega^2 = \omega_p^2(1 + 3k^2\lambda_D^2) \longrightarrow$ structure agrees with fluid mdl.

$\hookrightarrow$ contrast fluid

— Distribution function determines equation of state

i.e. # 3 $\leftrightarrow \int v^4 \langle f \rangle$

Contrast K+T : $p = p_0 (\rho/\rho_0)^\gamma \qquad \gamma = 3$

$\gamma = 3 \leftrightarrow$ Maxwellian

— Structure of dispersion relation identical to warm fluid model
$\Longleftrightarrow k v_{Th} < \omega$

$\rightarrow \epsilon_{IM}$.

$$\epsilon_{IM} = -\pi \frac{\omega_p^2}{k|k|} \left.\frac{\partial \langle f \rangle}{\partial v}\right|_{\omega/k}$$

$$Q = \omega \, \epsilon_{IM} \left(|E|^2 / 8\pi \right)$$ $\rightarrow$ dissipated energy

$\Rightarrow$

$$Q = -\omega_k \frac{\pi \omega_p^2}{k|k|} \left.\frac{\partial \langle f \rangle}{\partial v}\right|_{\omega_k/k} |E|^2/8\pi$$

Now,
$$\frac{\partial W_{\underline{k}}}{\partial t} + \nabla \cdot \underline{S}_{\underline{k}} + Q_{\underline{k}} = 0$$

$$\Rightarrow \gamma_{\underline{k}} = - Q_{\underline{k}} / W_{\underline{k}} \qquad W_{\underline{k}} = \omega_{\underline{k}} \left.\frac{\partial \epsilon_r}{\partial \omega}\right|_{\omega_{\underline{k}}} \frac{|E|^2}{8\pi}$$

$$\therefore \quad \gamma_{\underline{k}} = \left(\frac{\pi \omega_p^2}{k|k|} \left.\frac{\partial \langle f \rangle}{\partial v}\right|_{\frac{\omega_k}{k}} \right) \Big/ \left(\left.\frac{\partial \epsilon_r}{\partial \omega}\right|_{\omega_{\underline{k}}} \right)$$

Alternatively:

$$\epsilon = \epsilon_R(\underline{k}, \omega) + i\,\epsilon_{IM}(\underline{k}, \omega)$$

$$\omega = \omega_{\underline{k}} + i\gamma_{\underline{k}} \qquad \gamma \ll \omega_{\underline{k}}$$

$$\epsilon = \epsilon_R(k, \omega_{\underline{k}} + i\gamma_{\underline{k}}) + i\,\epsilon_{IM}(k, \omega_{\underline{k}})$$

$$\cong \epsilon_R(k, \omega_{\underline{k}}) + i\gamma_{\underline{k}} \left.\frac{\partial \epsilon_R}{\partial \omega}\right|_{\omega_{\underline{k}}} + i\,\epsilon_{IM}(k, \omega_{\underline{k}})$$

$$\gamma_{\underline{k}} = - \epsilon_{IM}(k, \omega_{\underline{k}}) \Big/ (\partial \epsilon_R / \partial \omega)\Big|_{\omega_{\underline{k}}}$$

agrees above.

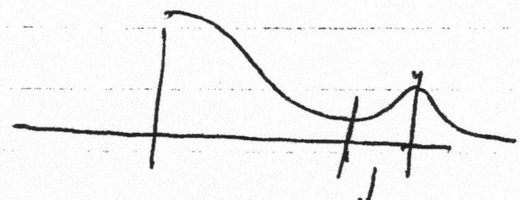
Thus $\rightarrow$ $\partial\langle f\rangle/\partial v\big|_{\omega_R/k} < 0$

$\Rightarrow$ damping (Landau damping)

$\rightarrow$ $\partial\langle f\rangle/\partial v\big|_{\omega/k} > 0$

$\Rightarrow$ growth

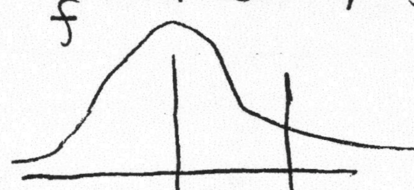
i.e. 'Bump on Tail'



$\omega/k \sim V$ grows as $\partial\langle f\rangle/\partial v > 0$

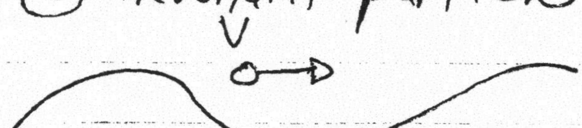
$\rightarrow$ Physics of Landau Damping

Consider



$\rightarrow$ Landau damping occurs due wave particle resonance $\omega/k \sim V$

$\rightarrow$ intuitively, consider wave interaction with ⊙ resonant particle



$\omega/k = v_{ph}$

Resonant particle 'sees' ⊙ DC field

$$\frac{dv}{dt} = \frac{q}{m} E \cos(kx - \omega t)$$

$$= \frac{q}{m} E \cos\left(k\left(x - v_{ph} t\right)\right)$$

if boost to frame at particle velocity V

$$x' = x - Vt$$
$$v' = v - V$$
$$a' = a$$

$\Rightarrow$

$$\frac{dv}{dt} = \frac{q}{m} E \cos\left(k\left(x + (V - v_{ph})t\right)\right)$$

$\therefore$

- secular (in time) interaction at $V \sim v_{ph}$ resonance

- $v \lesssim \omega/k \Rightarrow$ wave accelerates particles, loses energy

  $v \gtrsim \omega/k \Rightarrow$ wave decelerates particles, gains energy

Q = # accelerated − # decelerated

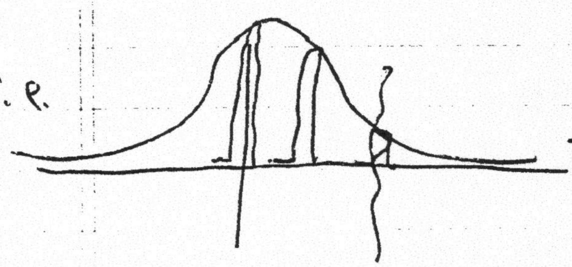
$$\sim \left(\partial f/\partial v\right)\Big|_{\omega/k}$$

▷ Quantitatively :

— as $Q = \langle \underline{E}^* \cdot \underline{J} \rangle$

seek $\bar{q} = \langle q v \underline{E} \rangle$ ⇒ time averaged work on resonant 'beam'

i.e.



⇒ plasma distribution as superposition of beams

then $Q = \int dv \, \bar{q}$

— $V = V_0 + \delta V$ ↳ perturbations induced by wave

$x = x_0 + \delta x$

so

$$\frac{d\,\delta V}{dt} = \frac{q}{m} E \Big|_{x_0, V_0}$$

$$\frac{d\delta x}{dt} = \delta V$$

$$\bar{q} = q \langle V E \rangle$$

$V = V_0 + \delta V$

$E = E(t, x = x_0 + \delta x)$

$\cong E(t, x_0) + \delta x \, \partial E/\partial x \big|_{x_0}$

$$\bar{\mathcal{I}} = q\left\langle (\underset{\text{DC}}{V_0} + \underset{\text{osc}}{\delta V})\left(\underset{\text{osc}}{E(t,x_0)} + \underset{\text{both osc.}}{\delta x\, \partial E/\partial x\big|_{x_0,t}}\right)\right\rangle$$

$$\bar{\mathcal{I}} = q V_0 \left\langle \delta x\, \partial E/\partial x\big|_{x_0,t}\right\rangle + q\left\langle \delta V\, E(t,x_0)\right\rangle$$

Now, $\dfrac{d\delta V}{dt} = \dfrac{q}{m} E(t,x_0)$ $\qquad x_0 = x_0' + V_0 t$

$$= \frac{q}{m} E_0\, e^{ikx_0'}\, e^{ik(V_0-\omega/k)t}\, e^{\delta t}$$

$x_0' = 0$ (convenience)

$\omega/k = V_{ph}$

$\delta > 0 \Rightarrow \delta V \to 0$ as $t \to -\infty$

$\therefore$

$$\frac{d\delta V}{dt} = \frac{q}{m} E_0 \exp\left(ik\left(V_0 - \omega/k - i\delta\right)t\right)$$

$$\delta V = \frac{q}{m}\, \frac{E_0\, e^{ik(V_0-\omega/k-i\delta)t}}{i\left(k(V_0-V_{ph})-i\delta\right)}\Bigg|_{-\infty}^{t}$$

$$\Rightarrow \delta V = \frac{q}{m} E(t,x_0)\Big/ ik\left(V_0 - V_{ph}\right) + \delta$$

$$\delta x = \frac{q}{m} E(t,x_0)\Big/\left(ik(V_0-V_{ph})+\delta\right)^2$$

Thus

$$\bar{q} = q v_0 \left\langle \delta x \frac{\partial E}{\partial x} \right\rangle + q \left\langle \delta v \, E \right\rangle$$

$$= q v_0 \left\langle -ik \, E^*(t, x_0) \, \frac{q}{m} \frac{E(t, x_0)}{(ik(v_0 - v_p) + \delta)^2} \right\rangle$$

$$+ q \left\langle E^*(t, x_0) \, \frac{q}{m} \frac{E(t, x_0)}{(ik(v_0 - v_p) + \delta)} \right\rangle$$

note: $E^* E$ gives DC beat

$\Rightarrow$

$$\bar{q} = \frac{d}{dv_0} \left\{ \frac{q^2}{2m} |E|^2 \frac{v_0}{ik(v_0 - v_p) + \delta} \right\}$$

$$= \frac{d}{dv_0} \left\{ \frac{q^2}{2m} |E|^2 \frac{-i v_0}{k(v_0 - v_p) - i\delta} \right\}$$

{ note: '2' from $\cos^2$

real part $\Rightarrow$

$$\bar{q} = \frac{d}{dv_0} \left\{ \frac{q^2}{2m} |E|^2 \frac{v_0 \pi}{|k|} \delta(v_0 - v_p) \right\}$$

$$Q = n\int dv_0\, \bar{q}(v_0)\, \langle f(v_0)\rangle$$

$$= \int dv_0\, \langle f(v_0)\rangle \frac{d}{dv_0}\left\{\frac{nq^2}{2m}|E|^2 \frac{v_0}{|k|}\pi\, \delta(v_0 - v_{ph})\right\}$$

$$= -\frac{\pi\omega_p^2}{|k|}\frac{\omega}{k}\frac{\partial\langle f(v)\rangle}{\partial v}\bigg|_{\omega/k}\left(|E|^2/8\pi\right)$$

$\Rightarrow$

$$Q = -\pi\frac{\omega_p^2}{|k|}\frac{\omega}{k}\frac{\partial\langle f\rangle}{\partial v}\bigg|_{\omega/k}\left(|E|^2/8\pi\right)$$

→ agrees with previous result

→ establishes Landau damping mechanism as collisionless heating, due secularity at wave-particle resonance.

→ Fate of energy:

$$\frac{\partial W_k}{\partial t} + \cancel{\nabla\cdot S_k} + Q_k = 0$$

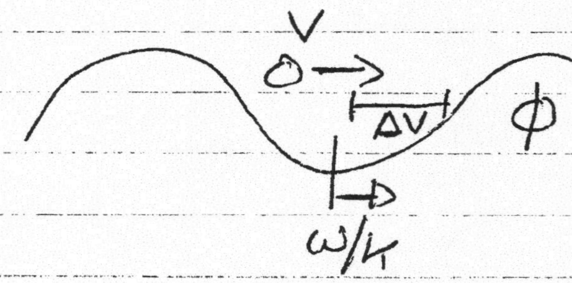
$$\frac{\partial W_k}{\partial t} = -Q_k \qquad \Rightarrow \text{L.D.} \leftrightarrow \text{wave energy dissipated}$$

at clearly resonant particles heated

so $$\frac{\partial \, RPKED}{\partial t} + \frac{\partial}{\partial t} W_k = 0$$

∴ Landau damping heats resonant piece of distribution at expense of wave energy.

→ Clearly, linear theory of Landau damping only valid for times less than bounce time in trough of wave:



$$\Delta V \sim \sqrt{q\phi/m}$$

$$1/\tau_b = k\,\Delta V$$

Then $\gamma_k = \gamma_k^{(0)}$ for $t < \tau_b$, only.